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AN ESTIMATE FOR THE GAUSS CURVATURE OF MINIMAL SURFACES IN $\mathbf{R}^m$ WHOSE GAUSS MAP OMITS A SET OF HYPERPLANES

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1. Introduction

The purpose of this paper is to prove the following theorem.

Theorem 1.1 (Main Theorem). *Let $x : M \to \mathbf{R}^m$ be a minimal surface immersed in $\mathbf{R}^m$. Suppose that its generalized Gauss map g omits more than $\frac{m(m+1)}{2}$ hyperplanes in $\mathbf{P}^{m-1}(\mathbf{C})$, located in general position. Then there exists a constant C, depending on the set of omitted hyperplanes, but not the surface, such that*

$$|K(p)|^{1/2} d(p) \leq C, \tag{1}$$

where $K(p)$ is the Gauss curvature of the surface at p, and $d(p)$ is the geodesic distance from p to the boundary of M.

This theorem provides a considerable sharpening of an earlier result of the same type:

Theorem 1.2 (Osserman [12]). *An inequality of the form (1) holds for all minimal surfaces in $\mathbf{R}^m$ whose Gauss map omits a neighborhood of some hyperplane in $\mathbf{P}^{m-1}(\mathbf{C})$.*

Also, Theorem 1.1 implies the earlier result:

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Theorem 1.3 (Ru [15]). *Let $x : M \to \mathbf{R}^m$ be a complete minimal surface immersed in $\mathbf{R}^m$. Suppose that its generalized Gauss map g omits more than $\frac{m(m+1)}{2}$ hyperplanes in $\mathbf{P}^{m-1}(\mathbf{C})$, located in general position. Then g is constant and the minimal surface must be a plane.*

In fact, given any point p on a complete surface satisfying the hypotheses, inequality (1) must hold with $d(p)$ arbitrarily large, so that $K(p) = 0$. But a minimal surface in $\mathbf{R}^m$ with $K \equiv 0$ must lie on a plane (see [10]) and hence its Gauss map g is constant.

Theorem 1.3 had been proved earlier by Fujimoto [5] in the case where the Gauss map g was assumed nondegenerate. Fujimoto (see [7]) also showed that the number $m(m+1)/2$ was optimal in that for every odd dimension m, there exist complete minimal surfaces whose Gauss map omits $m(m+1)/2$ hyperplanes in general position. It follows that Theorem 1.1 is also an optimal result of its type, since with any smaller number of omitted hyperplanes, a universal inequality of the form (1) cannot be valid, at least in odd dimensions.

When $m = 3$, we may consider the classical Gauss map into the unit sphere. Fujimoto [4] showed that an inequality of type (1) holds whenever the Gauss map omits 5 given points. Later [6] he obtained an expression for C that makes more explicit its dependence on the given points. Ros [14] gave a different proof which does not yield an explicit value for the constant C, but allows the extension to higher dimension that we give here.

2. Some theorems and lemmas

In this section, we recall some results which will be used later.

We first recall the following construction theorem of minimal surfaces.

Theorem 2.1 (see [3]). *Let M be an open Riemann surface and let $\omega_1, \omega_2, \dots, \omega_m$ be holomorphic forms on M having no common zero and no real periods, and locally satisfying the identity*

$$f_1^2 + f_2^2 + \cdots + f_m^2 = 0$$

for holomorphic functions f_i with $\omega_i = f_i dz$. Set

$$x_i = 2Re \int_{z_0}^{z} \omega_i,$$

for an arbitrary fixed point z_0 *of* M*. Then the surface* $x = (x_1, \ldots, x_m) : M \to \mathbf{R}^m$ *is a minimal surface immersed in* $\mathbf{R}^m$ *such that the Gauss map is the map* $g = [\omega_1 : \cdots : \omega_m] : M \to Q_{m-2}(\mathbf{C})$ *and the induced metric is given by*

$$ds^2 = 2(|\omega_1|^2 + \cdots + |\omega_m|^2).$$

The following is the general version of Hurwitz's theorem:

Theorem 2.2 (Hurwitz's theorem). *Let* $f_j : M \to N$ *be a sequence of holomorphic maps between two connected complex manifolds converging uniformly on every compact subset of* M *to a holomorphic map* f*. If the image of each map* f_j *misses a divisor* D *of* N*, then either the image of* f *misses* D *or it lies entirely in* D*.*

Proof. Assume first that $D = \{z | g(z) = 0\}$ for some holomorphic function g. Then $g \circ f_j$ is a sequence of holomorphic functions converging to the holomorphic function $g \circ f$. Since $g \circ f_j$ is non-vanishing, by the classical Hurwitz theorem the limit function is either identically zero or non-vanishing. In other words the image of f either lies entirely in D or misses D completely.

In the general case, if f does not miss D entirely, then there exist a point q in D and a point p in M such that $f(p) = q$. There exist a neighborhood U of q and a holomorphic function g on U so that $D \cap U = \{z | g(z) = 0\}$. Applying the previous argument to the restriction of the sequence of maps to the open set $V = f^{-1}(U)$ in U, we conclude that $f(V)$ is contained in $D \cap U$. Since M is connected, the principle of analytic continuation implies that the image $f(M)$ is contained in D.

q.e.d.

Lemma 2.1. *Let* D_r *be the disk of radius* $r, 0 < r < 1$*, and let* R *be the hyperbolic radius of* D_r *in the unit disc. Let*

$$ds^2 = \lambda(z)^2 |dz|^2$$

be any conformal metric on D_r *with the property that the geodesic distance from* $z = 0$ *to* $|z| = r$ *is greater than or equal to* R*. If the Gauss curvature* K *of the metric* ds^2 *satisfies*

$$-1 \le K \le 0,$$

then the distance of any point to the origin in the metric ds^2 *is greater than or equal to the hyperbolic distance.*

Remark 2.1. The hyperbolic metric in the unit disk is given by

$$d\hat{s}^2 = \hat{\lambda}(z)^2|dz|^2, \qquad \hat{\lambda}(z) = \frac{2}{1-|z|^2},$$

and has curvature $\hat{K} \equiv -1$. The relation between the quantities R and r is therefore given by

$$R = \int_0^r \hat{\lambda}(z)|dz| = \int_0^r \frac{2}{1-t^2}dt = \log\frac{1+r}{1-r},$$

and the conclusion of Lemma 2.1 is that

$$\rho(z) \geq \hat{\rho}(z) = \log\frac{1+|z|}{1-|z|},$$

where ρ and $\hat{\rho}$ represent the distances from the point z to the origin in the metric ds^2 and the hyperbolic metric, respectively.

Remark 2.2. Lemma 2.1 and its proof are basically geometric reformulations of Lemma 6 of Ros[14]. The lemma may be viewed as a kind of dual to the Ahlfors form of the Schwarz-Pick lemma [1].

Proof of Lemma 2.1. Note first that in the relation above between R and r, we have

$$\frac{dR}{dr} = \frac{2}{1-r^2} > 0,$$

and we may solve for r in terms of R:

$$r = \frac{e^R - 1}{e^R + 1}, \tag{2}$$

or in general

$$|z| = \frac{e^{\hat{\rho}(z)} - 1}{e^{\hat{\rho}(z)} + 1}, \tag{3}$$

where the right-hand side is monotone increasing in $\hat{\rho}(z)$. We may apply a comparison theorem of Greene and Wu ([9, Prop. 2.1, p.26]) to the two metrics, ds^2 and the hyperbolic metric $d\hat{s}^2$, on the disk $|z| \leq r$. The comparison theorem states that for any smooth monotone increasing function f, one has

$$\triangle(f \circ \rho) \leq \hat{\triangle}(f \circ \hat{\rho}),$$

where ρ and $\hat{\rho}$ are the distances to the origin in the metrics ds^2 and $d\hat{s}^2$ respectively, $\triangle$ and $\hat{\triangle}$ are the Laplacians with respect to the two metrics, and the two sides are evaluated at points of the same level sets of the two metrics, i.e., $\rho = c$ on the left and $\hat{\rho} = c$ on the right, provided in two dimensions that the Gauss curvatures K and $\hat{K}$ satisfy $0 \geq K \geq \hat{K}$, with a similar condition on Ricci curvature in higher dimension. In our case we have $0 \geq K \geq -1 = \hat{K}$, and so we may apply the theorem. We note that the function

$$\log|z| = \log \frac{e^{\hat{\rho}(z)} - 1}{e^{\hat{\rho}(z)} + 1}$$

is harmonic with respect to z and is therefore also harmonic with respect to any conformal metric on $0 < |z| < 1$. In other words, if we set

$$f(t) = \log \frac{e^t - 1}{e^t + 1},$$

we have

$$\hat{\triangle}(f \circ \hat{\rho}) \equiv 0$$

for $0 < |z| < 1$. Since f is monotone increasing, we may apply the Greene-Wu comparison theorem to conclude that

$$\triangle(f \circ \rho) \leq 0$$

for $0 < |z| < r$, i.e., $f \circ \rho$ is superharmonic. For z near 0, we have $\rho(z) \sim \lambda(0)|z|$, and we may apply the minimum principle to the function

$$f \circ \rho - \log|z| = \log \frac{1}{|z|} \frac{e^{\rho(z)} - 1}{e^{\rho(z)} + 1},$$

which is superharmonic in $0 < |z| < r$ and bounded near the origin, to conclude that it takes on its minimum on the boundary $|z| = r$. But since $\rho(z) \geq R$ on $|z| = r$, we have for $|z| < r$ that

$$\log \frac{1}{|z|} \frac{e^{\rho(z)} - 1}{e^{\rho(z)} + 1} \geq \log \frac{1}{r} \frac{e^R - 1}{e^R + 1} = 0,$$

by (2). Hence

$$\frac{e^{\rho(z)} - 1}{e^{\rho(z)} + 1} \geq |z| = \frac{e^{\hat{\rho}(z)} - 1}{e^{\hat{\rho}(z)} + 1},$$

by (3), which implies $\rho(z) \geq \hat{\rho}(z)$, proving the lemma. q.e.d.

As an application of Lemma 2.1, we have the following lemma:

Lemma 2.2. *Let ds_n^2 be a sequence of conformal metrics on the unit disk D whose curvatures satisfy $-1 \le K_n \le 0$. Suppose that D is a geodesic disk of radius R_n with respect to the metric ds_n^2, where $R_n \to \infty$, and that the metrics ds_n^2 converge, uniformly on compact sets, to a metric ds^2. Then all distances to the origin with respect to ds^2 are greater than or equal to the corresponding hyperbolic distances in D. In particular, ds^2 is complete.*

Proof. For any point z in D, let $\rho_n(z)$ be the distance from 0 to z in the metric ds_n^2, and let $\rho(z)$ be the distance in the limit metric ds^2. Let $|z| = r_n$ be the circle in D of hyperbolic radius R_n. Explicitly, by Remark 2.1 above,

$$R_n = \log \frac{1 + r_n}{1 - r_n}.$$

If we make the change of parameter $w = r_n z$, we may apply Lemma 2.1 to the induced metric in $|w| < r_n$ and conclude that

$$\rho_n(z) \ge \log \frac{1 + |w|}{1 - |w|} = \log \frac{1 + r_n|z|}{1 - r_n|z|}.$$

As $n \to \infty$ we have $R_n \to \infty$ and $r_n \to 1$. Hence, by uniform convergence on compact sets, we have

$$\rho(z) = \lim_{n\to\infty} \rho_n(z) \ge \lim_{r_n \to 1} \log \frac{1 + r_n|z|}{1 - r_n|z|} = \log \frac{1 + |z|}{1 - |z|},$$

which proves the lemma. q.e.d.

Note. Although we shall not make use of it, we remark that Lemma 2.1 also implies another dual form of the Ahlfors-Schwarz-Pick lemma, closer in form to the original:

Lemma 2.3. *Let S be a simply-connected surface with a complete metric ds^2 whose Gauss curvature satisfies $-1 \le K \le 0$. If S is mapped conformally onto the unit disc, then the distance between any two points of S is greater than or equal to the hyperbolic distance between the corresponding points in the disk.*

Proof. Given two points p, q of S, we may map p onto the origin, and let z be the image of the point q. Then the distance between p and q on S is given by $\rho(z)$ in terms of the pull-back of the metric on S onto the disk. For any r such that $|z| < r < 1$, let $\hat{\rho}(z)$ be the hyperbolic distance from 0 to z, and let $\rho_r(w)$ be the pullback of the metric on S

to $|w| < r$ under the map $z = w/r$. Then, since S is complete, we may apply Lemma 2.1 to conclude that

$$\hat{\rho}(z) \leq \rho_r(w) = \rho_r(rz).$$

But as $r \to 1, \rho_r(rz) \to \rho(z)$, which proves the lemma. q.e.d.

Note that Lemma 2.3 combined with the standard Ahlfors-Schwarz-Pick lemma implies a generalization of Ahlfors' lemma due to Yau ([17]; see also Troyanov [16]): *Let S_1 be a simply-connected Riemann surface with a complete metric ds^2 whose Gauss curvature satisfies $-1 \leq K \leq 0$, and let S_2 be a Riemann surface with Gauss curvature bounded above by -1. Let $f : S_1 \to S_2$ be a holomorphic map. Then f is distance decreasing.*

We also need the following more precise version of Theorem 1.3; the proof follows exactly as in [15].

Theorem 2.3 *(cf. Ru [15]). Let $x : M \to \mathbf{R}^m$ be a complete minimal surface immersed in $\mathbf{R}^m$. Suppose that its generalized Gauss map g omits the hyperplanes $H_1, \ldots, H_q$ in $\mathbf{P}^{m-1}(\mathbf{C})$ and $g(M)$ is contained in some $\mathbf{P}(V)$, where V is a subspace of $\mathbf{C}^m$ of dimension k. Assume that $H_1 \cap \mathbf{P}(V), \ldots, H_q \cap \mathbf{P}(V)$ are in general position in $\mathbf{P}(V)$ and $q > k(k+1)/2$. Then g must be constant.*

The following theorem due to M. Green (see [8]) shows that the complement of $2m+1$ hyperplanes in general position in $\mathbf{P}^m(\mathbf{C})$ is complete Kobayashi hyperbolic.

Theorem 2.4. *Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbf{P}^m(\mathbf{C})$, located in general position. If $q \geq 2m+1$, then $X = \mathbf{P}^m(\mathbf{C}) - \cup_{j=1}^q H_j$ is complete hyperbolic and hyperbolically imbedded in $\mathbf{P}^m(\mathbf{C})$. Hence, if $D \subset \mathbf{C}$ is the unit disc, and Φ is a subset of $Hol(D, X)$, then Φ is relatively locally compact in $Hol(D, \mathbf{P}^m(\mathbf{C}))$, i.e., given a sequence $\{f_n\}$ in Φ there exists a subsequence which converges uniformly on every compact subset of D to an element of $Hol(D, \mathbf{P}^m(\mathbf{C}))$.*

For the notions of "complete Kobayashi hyperbolicity" and "hyperbolically imbedded in $\mathbf{P}^m(\mathbf{C})$", see Lang [11].

Before going to the next section, we recall here a standard definition.

Definition 2.1. Let $f : M \to \mathbf{P}^n(\mathbf{C})$ be a holomorphic map. Let $p \in M$. **A local reduced representation of f around p** is a holomorphic map $\tilde{f} : U \to \mathbf{C}^{n+1} - \{\mathbf{0}\}$, such that $\mathbf{P}(\tilde{f}) = f$, where U is a neighborhood of p, and $\mathbf{P}$ is the projection map of $\mathbf{C}^{n+1} - \{0\}$ onto $\mathbf{P}^n(\mathbf{C})$.

3. Proof of the Main Theorem

Let $x : M \to \mathbf{R}^m$ be a minimal surface, where M is a connected, oriented, real-dimension 2 manifold without boundary, and

$$x = (x_1, \ldots, x_m)$$

is an immersion. Then M is a Riemann surface in the induced structure defined by local isothermal coordinates (u, v). The generalized Gauss map of the minimal surface,

$$g = [\frac{\partial x_1}{\partial z} : \cdots : \frac{\partial x_m}{\partial z}] : M \to Q_{m-2}(\mathbf{C}) \subset \mathbf{P}^{m-1}(\mathbf{C})$$

is a holomorphic map, where $z = u + iv$. The metric ds^2 on M, induced from the standard metric in $\mathbf{R}^m$, is $ds^2 = \sum_{j=1}^m |\frac{\partial x_j}{\partial z}|^2 dz d\bar{z}$, and the Gauss curvature K is given by ([10, p.37])

$$K = -4\frac{|\tilde{g} \wedge \tilde{g}'|^2}{|\tilde{g}|^6} = -4\frac{\sum_{j<k} |g_j g_k' - g_k g_j'|^2}{(\sum_{j=1}^m |g_j|^2)^3}, \tag{4}$$

where $\tilde{g} = (g_1, \ldots, g_m), g_j = \frac{\partial x_j}{\partial z}, 1 \le j \le m$.

We will need the following lemma:

Lemma 3.1. *Let M be a Riemann surface. Let $f_n : M \to \mathbf{P}^m(\mathbf{C})$ be a sequence of holomorphic maps converging uniformly on every compact subset of M to a holomorphic map $f : M \to \mathbf{P}^m(\mathbf{C})$. Given $\mathbf{a}, \mathbf{b} \in \mathbf{P}^m(\mathbf{C}^*)$, let $f_{\mathbf{a},\mathbf{b}}$ be the meromorphic function (called coordinate function) defined by*

$$f_{\mathbf{a},\mathbf{b}}|_U = \frac{\alpha(\tilde{f})}{\beta(\tilde{f})},$$

where $\tilde{f}$ is a reduced representation of f on U, and $\alpha, \beta \in \mathbf{C}^{m+1^*}$ such that $\mathbf{a} = \mathbf{P}(\alpha), \mathbf{b} = \mathbf{P}(\beta)$. Assume that $\beta(\tilde{f}) \not\equiv 0$ on some U (i.e., the image of f is not contained in the hyperplane defined by $\mathbf{b}$). Let $p \in M$ be such that $\beta(\tilde{f})(p) \neq 0$, and U_p be a neighborhood of p such that $\beta(\tilde{f})(z) \neq 0$ for $z \in U_p$; then $\{f_{n_{\mathbf{a},\mathbf{b}}}\}$ converges uniformly on U_p to the meromorphic function $f_{\mathbf{a},\mathbf{b}}$.

Proof. Since the image of f is not contained in the hyperplane defined by $\mathbf{b}$, the image of f_n is also not contained in the hyperplane defined by $\mathbf{b}$ for n large enough. Since $\frac{\mathbf{a}(\mathbf{x})}{\mathbf{b}(\mathbf{x})}$ is a rational function on $\mathbf{P}^m(\mathbf{C})$ and f_n converges uniformly on every compact subset of M to f, the composition functions also converge compactly. This concludes the proof. q.e.d.

Lemma 3.2. *Let $x^{(n)} = (x_1^{(n)}, \ldots, x_m^{(n)}) : M \to \mathbf{R}^m$ be a sequence of minimal immersions, and $g^{(n)} : M \to Q_{m-2}(\mathbf{C}) \subset \mathbf{P}^{m-1}(\mathbf{C})$ the sequence of their (generalized) Gauss maps. Suppose that $\{g^{(n)}\}$ converges uniformly on every compact subset of M to a non-constant holomorphic map $g : M \to Q_{m-2}(\mathbf{C}) \subset \mathbf{P}^{m-1}(\mathbf{C})$ and that there is some $p_0 \in M$ such that for each $j, 1 \leq j \leq m$, $\{x_j^{(n)}(p_0)\}$ converges. Assume also that $\{|K_n|\}$ is uniformly bounded, where K_n is the Gauss curvature of the minimal surface $x^{(n)}$. Then*

(i) either a subsequence $\{K_{n'}\}$ of $\{K_n\}$ converges to zero or

(ii) a subsequence $\{x^{(n')}\}$ of $\{x^{(n)}\}$ converges to a minimal immersion, $x : M \to \mathbf{R}^m$, whose Gauss map is g.

Proof. By assumption, g is not constant and we may assume that $|K_n| \leq 1$ in M, for each $n \in \mathbf{N}$. For every point $p \in M$ let (U_p, z) be a complex local coordinate centered at p. Let $\tilde{g}^{(n)} = (g_1^{(n)}, \ldots, g_m^{(n)})$ where $g_i^{(n)} = \frac{\partial x_i^{(n)}}{\partial z}, 1 \leq i \leq m$, and let $\tilde{g} = (g_1, \ldots, g_m)$ be a local reduced representation of g on U_p. Since some $g_i(z)$ is non-zero for each z, we know that $g(M)$ is not contained in some coordinate hyperplane. Without loss of generality, we assume that $g(M)$ is not contained in the first coordinate hyperplane $H_1 = \{[y_1 : \cdots : y_m] \in \mathbf{P}^{m-1}(\mathbf{C}) | y_1 = 0\}$. Let

$$M_1 = \{p \in M | g(p) \notin H_1, \tilde{g}(p) \wedge \tilde{g}'(p) \neq 0\}.$$

Note that $M - M_1$ is a discrete set: namely, it consists of the zeros of g_1 (which are isolated, since $g(M) \not\subset H_1$, which is equivalent to $g_1 \not\equiv 0$) together with the common zeros of the components of $\tilde{g} \wedge \tilde{g}'$, which are the holomorphic functions $g_j g_k' - g_k g_j'$. In particular,

$$g_1 g_k' - g_k g_1' = g_1^2 (\frac{g_k}{g_1})',$$

so that $\tilde{g} \wedge \tilde{g}' \equiv 0$ implies that $g_k / g_1 = c_k$, a constant for each k, so that $\tilde{g} = g_1(1, c_2, \ldots, c_m)$ and the map g would be constant, contrary to assumption. Thus, the zeros of $\tilde{g} \wedge \tilde{g}'$ are isolated and the points of $M - M_1$ are also isolated.

Let $p \in M_1$. Since $g(p) \notin H_1$, there is a neighborhood U_p of p such that $g(z) \notin H_1$, and $g^{(n)}(z) \notin H_1$ for n large enough and every $z \in U_p$. Choosing U_p sufficiently small, we have that $g_2/g_1, \dots, g_n/g_1$ are holomorphic and

$$4\frac{|\tilde{g} \wedge \tilde{g}'|^2/|g_1|^4}{(1+\sum_{j=2}^m |g_j/g_1|^2)^3} = 4\frac{\sum_{j<k} |\frac{g_j}{g_1}(\frac{g_k}{g_1})' - \frac{g_k}{g_1}(\frac{g_j}{g_1})'|^2}{(1+\sum_{j=2}^m |g_j/g_1|^2)^3} \geq 2c_1,$$

in U_p, where c_1 is some positive constant. Since $g^{(n)} \to g$ uniformly, by Lemma 3.1, $\{g_j^{(n)}/g_1^{(n)}\}$ converges uniformly to g_j/g_1 on U_p, $1 \leq j \leq m$. So we have

$$4\frac{\sum_{j<k} |\frac{g_j^{(n)}}{g_1^{(n)}}(\frac{g_k^{(n)}}{g_1^{(n)}})' - \frac{g_k^{(n)}}{g_1^{(n)}}(\frac{g_j^{(n)}}{g_1^{(n)}})'|^2}{(1+\sum_{j=2}^m |g_j^{(n)}/g_1^{(n)}|^2)^3} \geq c_1,$$

in U_p, and by (4),

$$\frac{c_1}{|g_1^{(n)}|^2} \leq 4\frac{\sum_{l<k} |\frac{g_l^{(n)}}{g_1^{(n)}}(\frac{g_k^{(n)}}{g_1^{(n)}})' - \frac{g_k^{(n)}}{g_1^{(n)}}(\frac{g_l^{(n)}}{g_1^{(n)}})'|^2}{|g_1^{(n)}|^2(1+\sum_{j=2}^m |g_j^{(n)}/g_1^{(n)}|^2)^3} = |K_n| \leq 1,$$

in U_p. Therefore

$$c_1 \leq |g_1^{(n)}|^2$$

in U_p, for large n. Then $\{g_1^{(n)}\}$ is relatively compact in $\mathcal{M}(U_p)$. Noticing that $M - M_1$ is discrete, by taking a subsequence, if necessary, we can assume that the globally defined holomorphic 1-forms $\{g_1^{(n)}dz\}$ converge on M_1, to a holomorphic 1-form $h_1 dz$ or to infinity, uniformly on every compact subset of M_1. We consider each case below:

Case 1. $\{g_1^{(n)}dz\}$ converges to infinity uniformly on every compact subset of M_1.

For $p \in M_1$, we have, by (4),

$$(5) \quad K_n(p) = -4\frac{\sum_{j<k} |\frac{g_j^{(n)}(p)}{g_1^{(n)}(p)}(\frac{g_k^{(n)}}{g_1^{(n)}})'(p) - \frac{g_k^{(n)}(p)}{g_1^{(n)}(p)}(\frac{g_j^{(n)}}{g_1^{(n)}})'(p)|^2}{|g_1^{(n)}(p)|^2(1+\sum_{j=2}^m |g_j^{(n)}(p)/g_1^{(n)}(p)|^2)^3} \to 0.$$

Let p be a point such that $p \notin M_1$ but also $g(p) \notin H_1$; then in a small disc of U_p, $D(2\epsilon)$, $g^{(n)}(z) \notin H_1$ for n large enough, $z \in D(2\epsilon)$. This means that $g_1^{(n)}$ is non-vanishing on $D(2\epsilon)$ and $g_1^{(n)}$ converges to infinity on $\partial D(\epsilon)$. From the maximum principle we conclude that $\{g_1^{(n)}\}$ converges to infinity on $D(\epsilon)$. Therefore we again have $K_n(p) \to 0$ by (4).

Finally suppose that $g(p) \in H_1$, i.e., $g_1(p) = 0$. Since $g(p)$ is not contained in some coordinate hyperplane, we assume that $g(p) \notin H_2$, where H_2 is the second coordinate hyperplane, $H_2 = \{[y_1 : \cdots : y_n] \in \mathbf{P}^{n-1}(\mathbf{C}) | y_2 = 0\}$. Therefore, on a small disc, $D(2\epsilon)$, $g^{(n)}(z) \notin H_2$ for n large enough, i.e., $g_2^{(n)}(z) \neq 0$, for $z \in D(2\epsilon)$, and $g_1^{(n)}$, g_1 have no zeros on a neighborhood of $\partial D(\epsilon)$ for n large enough. By Lemma 3.1, $\{\frac{g_2^{(n)}}{g_1^{(n)}}\}$, as a sequence of non-vanishing holomorphic functions, converges uniformly on $\partial D(\epsilon)$. Clearly, $\{\frac{g_2^{(n)}}{g_1^{(n)}} g_1^{(n)}\}$ converges uniformly to infinity on $\partial D(\epsilon)$, and therefore $g_2^{(n)}$ converges uniformly to infinity on $\partial D(\epsilon)$. Again from the maximum principle, we conclude that $g_2^{(n)}$ converges to infinity on $D(\epsilon)$. By (4), noticing that

$$|\tilde{g}^{(n)} \wedge \tilde{g}^{(n)\prime}|^2 / |g_2^{(n)}|^4 = \sum_{j<k} |\frac{g_j^{(n)}}{g_2^{(n)}} (\frac{g_k^{(n)}}{g_2^{(n)}})' - \frac{g_k^{(n)}}{g_2^{(n)}} (\frac{g_j^{(n)}}{g_2^{(n)}})'|^2,$$

we have

$$K_n(p) = -4 \frac{\sum_{j<k} |\frac{g_j^{(n)}}{g_2^{(n)}} (\frac{g_k^{(n)}}{g_2^{(n)}})' - \frac{g_k^{(n)}}{g_2^{(n)}} (\frac{g_j^{(n)}}{g_2^{(n)}})'|^2}{|g_2^{(n)}|^2 (\sum_{j=1}^n |g_j^{(n)}/g_2^{(n)}|^2)^3} \to 0.$$

Thus, we have proved that $K_n(p) \to 0$ for all $p \in M$. This corresponds to case (i) of the lemma.

Case 2. $\{g_1^{(n)} dz\}$ converges to a holomorphic 1-form, $h_1 dz$, on M_1.

Let $p \in M - M_1$. If $D(2\epsilon)$ is a small disc contained in U_p, as $\{g_1^{(n)}\} \to h_1$ uniformly on $\partial D(\epsilon)$ and $g_1^{(n)}$ are holomorphic, using the maximum principle, we see that $\{g_1^{(n)}\}$ is relatively compact on $D(\epsilon)$. Therefore $h_1 dz$ extends to a holomorphic 1-form on M and the global 1-forms $\{g_1^{(n)} dz\}$ converge to $h_1 dz$ on M.

We now prove that, for every integer $j, 2 \leq j \leq m$, the global 1-forms $\{g_j^{(n)} dz\}$ converge to a holomorphic form $h_j dz$ on M. Let $p \in M$ such that $g(p) \notin H_1$; then there is a neighborhood U_p of p such that g_1, $g_1^{(n)}$ have no zeros for n large enough, $z \in U_p$. Since $g_j^{(n)} = \frac{g_j^{(n)}}{g_1^{(n)}} g_1^{(n)}$, and by Lemma 3.1, $\{\frac{g_j^{(n)}}{g_1^{(n)}}\}$ converges uniformly on U_p, and $g_1^{(n)}$ also converges uniformly on U_p, $\{g_j^{(n)}\}$ must converge uniformly on U_p. For the points p such that $g(p) \in H_1$, if $D(2\epsilon) \subset U_p$ is small enough so that $g_1, g_1^{(n)}$ have no zeros on a neighborhood of $\partial D(\epsilon)$ for n large enough, then we just proved that $\{g_j^{(n)}\}$ is uniformly convergent on $\partial D(\epsilon)$. Since $g_j^{(n)}$ are holomorphic, by the maximum principle, we have that $g_j^{(n)}$ converges uniformly on $D(\epsilon)$. Therefore the globally defined holomorphic 1-forms $g_j^{(n)} dz$ converge to $\omega_j = h_j dz$, $1 \leq j \leq m$.

We now check that the conditions in Theorem 2.1 are satisfied. Obviously we only need check that $\omega_j, 1 \leq j \leq m$, have no common zero. Take an arbitrary point $p \in M$; since $\tilde{g} = (g_1, \ldots, g_m)$ is a reduced representation of g, there is some integer $1 \leq k \leq m$, such that $g_k(p) \neq 0$. So there is some neighborhood U_p, such that g_k is non-vanishing on U_p, i.e., $g(U_p)$ omits k-th coordinate hyperplane H_k. Since $g^{(n)}$ converges to g uniformly on every compact subset of M, for n large enough, $g^{(n)}(U_p)$ also omits H_k. So $g_k^{(n)}$ is non-vanishing on U_p. By Hurwitz's theorem, since $g_k^{(n)}$ converges uniformly on U_p to h_k, either h_k is non-vanishing on U_p or $h_k \equiv 0$ on U_p. If h_k is non-vanishing on U_p, then we are done. Otherwise, $h_k \equiv 0$ on U_p, so $g_k^{(n)} \to 0$ on U_p. Pick a point $q \in U_p$ such that $q \in M_1$, then, by (4),

$$K_n(q) = -4 \frac{|g_k(q)|^2}{|g_k^{(n)}(q)|^2} \frac{\sum_{i<j} |\frac{g_i^{(n)}(q)}{g_k^{(n)}(q)} (\frac{g_j^{(n)}}{g_k^{(n)}})'(q) - \frac{g_j^{(n)}(q)}{g_k^{(n)}(q)} (\frac{g_i^{(n)}}{g_k^{(n)}})'(q)|^2}{|g_k(q)|^2 (1 + \sum_{j=1, j\neq k}^m |g_j^{(n)}(q)/g_k^{(n)}(q)|^2)^3}.$$

But

$$\frac{\sum_{i<j} |\frac{g_i^{(n)}(q)}{g_k^{(n)}(q)} (\frac{g_j^{(n)}}{g_k^{(n)}})'(q) - \frac{g_j^{(n)}(q)}{g_k^{(n)}(q)} (\frac{g_i^{(n)}}{g_k^{(n)}})'(q))|^2}{|g_k(q)|^2 (1 + \sum_{j=2, j\neq k}^m |g_j^{(n)}(q)/g_k^{(n)}(q)|^2)^3} \to \frac{|\tilde{g} \wedge \tilde{g}'|^2}{|\tilde{g}|^6}(q) \neq 0,$$

and $g_k(q) \neq 0$, $g_k^{(n)}(q) \to 0$. So $|K_n(q)| \to \infty$, which contradicts the assumption that $\{|K_n|\}$ is uniformly bounded. Therefore, the conditions in Theorem 2.1 are satisfied. So they define a minimal surface $x : M \to \mathbf{R}^m$ whose Gauss map is g. q.e.d.

We now prove the main theorem.

Proof of the Main Theorem.
Suppose the theorem is not true. We will construct a nonflat complete minimal surface whose Gauss map omits a set of hyperplanes in general position, thus getting a contradiction with Theorem 2.3. So suppose the conclusion of the theorem is not true; then there is a sequence of (non complete) minimal surfaces $x^{(n)} : M_n \to \mathbf{R}^m$ and points $p_n \in M_n$ such that $|K_n(p_n)|d_n^2(p_n) \to \infty$, and such that the Gauss map $g^{(n)}$ of $x^{(n)}$ omits a fixed set of q hyperplanes in general position, with $q > m(m+1)/2$.

We claim that the surfaces M_n can be chosen so that

$$(6) \qquad K_n(p_n) = -1, \quad -4 \le K_n \le 0 \quad \textit{on } M_n \textit{ for all } n.$$

We now prove the claim. Without loss of generality, we can assume that M_n is a geodesic disk centered at p_n. Let

$$M_n' = \{p \in M_n : d_n(p, p_n) \le d_n(p_n)/2\}.$$

Then K_n is uniformly bounded on M_n' and $d_n'(p) =$ distance of p to the boundary of M_n' tends to zero as $p \to \partial M_n'$. Hence $|K_n(p)|(d_n'(p))^2$ has a maximum at a point p_n' interior to M_n'. Therefore

$$|K_n(p_n')|d_n'(p_n')^2 \ge |K_n(p_n)|d_n'(p_n)^2 = \frac{1}{4}|K_n(p_n)|d_n^2(p_n) \to \infty.$$

So we can replace the M_n by the M_n', with $|K_n(p_n')|d_n'(p_n')^2 \to \infty$. We rescale M_n' to make $K_n(p_n') = -1$. By the invariance under scaling of the quantity $K(p)d(p)^2$, we will have $d_n'(p_n') \to \infty$; here, without causing confusion, we use the same notation d_n' to denote the geodesic distance with respect to the rescaled metric. Again we can assume that M_n' is a geodesic disc centered at p_n', and let

$$M_n'' = \{p \in M_n' | d_n(p, p_n') < \frac{d_n'(p_n')}{2}\}.$$

Then $p \in M_n''$ implies that $d_n'(p) \ge \frac{d_n'(p_n')}{2}$ and

$$|K_n(p)|\frac{d_n'(p_n')^2}{4} \le |K_n(p)|d_n'(p)^2 \le |K_n(p_n')|d_n'(p_n')^2 = d_n'(p_n')^2.$$

Therefore $|K_n(p)| \le 4$ on M_n''. Furthermore,

$$d_n''(p_n') = d(p_n', \partial M_n'') = d_n'(p_n')/2 \to \infty.$$

This proves the claim.

By translations of $\mathbf{R}^m$ we can assume that $x^{(n)}(p_n) = \mathbf{0}$. We can also assume that M_n is simply connected, by taking its universal covering, if necessary. By the uniformization theorem, M_n is conformally equivalent to either the unit disc D or the complex plane $\mathbf{C}$, and we can suppose that p_n maps onto 0 for each n. But the case that M_n is conformally equivalent to $\mathbf{C}$ is impossible because the condition that $g^{(n)}$ misses more than $m(m+1)/2$ hyperplanes in general position in $\mathbf{P}^{m-1}(\mathbf{C})$, implies, by Picard's theorem, that $g^{(n)}$ is constant, so $K_n \equiv 0$, which contradicts the condition that $|K_n(0)| = 1$. So we have constructed a sequence of minimal surfaces, $x^{(n)} : D \to \mathbf{R}^m$, satisfying (6). Since, by Theorem 2.4, $\mathbf{P}^{m-1}(\mathbf{C})$ minus $2m-1$ hyperplanes is complete Kobayashi hyperbolic, and $m(m+1)/2 \ge 2m-1$, a subsequence of generalized Gauss maps $g^{(n)}$ of $x^{(n)}$ exists−without loss of generality we assume $g^{(n)}$ itself−such that $g^{(n)} : D \to \mathbf{P}^{m-1}(\mathbf{C})$ converges uniformly on every compact subset of D to a map $g : D \to \mathbf{P}^{m-1}(\mathbf{C})$.

We now claim that g is non-constant. Suppose not, i.e., g is a constant map, and g maps the disk D onto a single point P. Let H be any hyperplane not containing the point P, and let U, V be disjoint neighborhoods of H and P respectively. Let C be the constant in Theorem 1.2 such that

$$|K(p)|^{1/2} d(p) \le C$$

for any minimal surface in $\mathbf{R}^m$ whose Gauss map omits the neighborhood U of H, where p is a point of S and $d(p)$ is the geodesic distance of p to the boundary of S. Choose $r < 1$ such that the hyperbolic distance R of $z = 0$ to $|z| = r$ satisfies $R > C$. Since $g^{(n)}$ converges uniformly to g on $|z| \le r$, the image of $|z| = r$ lies in the neighborhood V of P for sufficiently large n, say $n \ge n_0$. It follows that for $n \ge n_0$, the image of the disk $|z| \le r$ under $g^{(n)}$ omits the neighborhood U of H and we may therefore apply the above inequality to conclude

$$|K_n(0)|^{1/2} d_n(r) \le C,$$

where $d_n(r)$ is the geodesic distance from the origin to the boundary of the surface $x^{(n)} : D(r) \to \mathbf{R}^m$. But $|K_n(0)| = 1$ for all n, and hence

$d_n(r) \leq C$ for $n \geq n_0$. On the other hand, we get a lower bound for $d_n(r)$ from Lemma 2.1. The surface $x^{(n)} : \{|z| < 1\} \to \mathbf{R}^m$ is a geodesic disk of radius R_n. If we reparametrize by $w = r_n z$ where the subset $\{w \mid |w| < r_n\}$ has hyperbolic radius R_n, then the circle $|z| = r$ corresponds to $|w| = r_n r$, and by Lemma 2.1, the distance in the surface metric from the origin to any point on the circle $|z| = r$, or equivalently, $|w| = r_n r$, is greater than or equal to the hyperbolic distance from 0 to $|w| = r_n r$. But as $n \to \infty$, $R_n \to \infty$ and $r_n \to 1$, so that the hyperbolic radius of $|w| = r_n r$ tends to the hyperbolic radius of $|w| = r$, which is R. Since by assumption $R > C$ we have for n sufficiently large that the surface distance from $z = 0$ to $|z| = r$ is greater than C, contradicting the earlier bound $d_n(r) \leq C$. Thus we conclude that the limit function g can not be constant.

Therefore the hypotheses of Lemma 3.2 are satisfied. Since $|K_n(0)| = 1$, the possibility (i) of Lemma 3.2 cannot happen. Thus, a subsequence $\{x^{(n')}\}$ of $\{x^{(n)}\}$ converges to a minimal immersion $x : D \to \mathbf{R}^m$, whose Gauss map is g. By (6) and Lemma 2.2, x is complete. By assumption, $g^{(n)}$ omits hyperplanes $H_1, \ldots, H_q$ in $\mathbf{P}^{m-1}(\mathbf{C})$, located in general position, $q > m(m+1)/2$. By Hurwitz's theorem(Theorem 2.2), either g omits these hyperplanes, or the image of g lies in some of these hyperplanes. Say $g(M) \subset \cap_{j=1}^k H_j = \mathbf{P}(V)$, where V is a subspace of $\mathbf{C}^m$ of dimension $m-k$, and $g : M \to \mathbf{P}(V)$ omits the hyperplanes $H_{k+1} \cap (\cap_{j=1}^k H_j), \ldots, H_q \cap (\cap_{j=1}^k H_j)$ in $\mathbf{P}(V)$. Since the hyperplanes $H_{k+1} \cap (\cap_{j=1}^k H_j), \ldots, H_q \cap (\cap_{j=1}^k H_j)$ in $\mathbf{P}(V)$ are still in general position in $\mathbf{P}(V)$ because $H_1, \ldots, H_q$ are in general position in $\mathbf{P}^{m-1}(\mathbf{C})$, and $q - k > m(m+1)/2 - k \geq (m-k)(m-k+1)/2$, it follows from Theorem 2.3 that g is constant. But we have just proved that g is not constant. This leads to a contradiction. Therefore the main theorem is proved.

q.e.d.

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